Advanced Algebra: Polynomial

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Section A Linear Algebra

1. We call λ_0 an **algebraic integer** iff it is a root of a monic polynomial with integer coefficients. Answer the following four questions:

   (a) Assume $r \in \mathbb{Q}$; then r is an algebraic integer iff $r \in \mathbb{Z}$.

   (b) Assume $d \in \mathbb{Z}$; prove that $\sqrt{d}$ is an algebraic integer.

   (c) Prove that any **root of unity** is an algebraic integer.

   (d) Assume λ_1, λ_2 are algebraic integers; prove that $-\lambda_1, \overline{\lambda_1}, \lambda_1\lambda_2$, and $\lambda_1 + \lambda_2$ are as well.

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2. Find all eigenvalues of the **circulant matrix**:

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$$

3. Find the determinant of the following **anti-circulant matrix**:

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ -a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ -a_{n-2} & -a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & -a_2 & -a_3 & \cdots & a_0 \end{bmatrix}$$

4. Find the inverse of the following matrix:

$$\begin{bmatrix} 1 & b & b & \cdots & b^{n-1} \\ & 1 & b & \cdots & b^{n-2} \\ & & 1 & \cdots & b^{n-3} \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}$$

5. Find the inverse of the following matrix, given $a \neq 1$ and $a \neq \frac{1}{1-n}$:

$$\begin{bmatrix} 1 & a & \cdots & a \\ a & 1 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & 1 \end{bmatrix}$$

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6. Assume $d(x) = (f(x), g(x))$. Prove that the solution space of $d(A)x = 0$ is the intersection of the solution spaces of $f(A)x = 0$ and $g(A)x = 0$.

7. Assume $f = f_1 f_2$ and $(f_1, f_2) = 1$. Prove that any solution to $f(A)x = 0$ can be uniquely represented as the sum of solutions to $f_1(A)x = 0$ and $f_2(A)x = 0$.

8. Assume $(f, g) = 1$. Prove that:

$$R(f(A)) + R(g(A)) = n + R(f(A)g(A)).$$

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9. Let n be an odd integer. Prove that:

$$(x+y)(y+z)(x+z) \mid (x+y+z)^n - x^n - y^n - z^n.$$

10. Suppose $(x^4 + x^3 + x^2 + x + 1) \mid (x^3 f_1(x^5) + x^2 f_2(x^5) + x f_3(x^5) + f_4(x^5))$, where $f_i(x)$ $(1 \leq i \leq 4)$ are polynomials with real coefficients. Prove that $f_i(1) = 0$ for $1 \leq i \leq 4$.

Section B Division with Remainder

1. Assume m and n are positive integers. Prove that

$$(x^m - 1, x^n - 1) = x^{(m,n)} - 1.$$

2. Assume integers $m, n, l \geq 0$. Prove that

$$x^2 + x + 1 \mid x^{3m} + x^{3n+1} + x^{3l+2}$$

3. Let $(f(x), g(x)) = 1$. Prove that:

$$(f(x)g(x), f(x) + g(x)) = 1.$$

4. Let $(f(x), g(x)) = d(x)$. Prove that for any positive integer n:

$$(f(x)^n, f(x)^{n-1}g(x), \ldots, g(x)^n) = d(x)^n.$$

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5. Let $f(x)$ be a non-zero polynomial with real coefficients. If $f(f(x)) = (f(x))^k$, where k is a fixed positive integer, find $f(x)$.

6. Let $k \geq 2$ be a positive integer, and let $f(x)$ be a non-zero polynomial of positive degree with real coefficients. If $f(x)$ satisfies $f(x^k) = (f(x))^k$, find $f(x)$.

7. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. If the n roots $x_1, x_2, \ldots, x_n$ are all non-zero, find a polynomial whose roots are $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$.

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8. **Chinese Remainder Theorem for Polynomials**

   Let $f_1(x), f_2(x), \ldots, f_k(x) \in K[x]$ be pairwise relatively prime polynomials (i.e., $(f_i, f_j) = 1$ for $i \neq j$). For any given polynomials $r_1(x), r_2(x), \ldots, r_k(x) \in K[x]$, there exists a polynomial $f(x) \in K[x]$ such that

$$\begin{aligned} f(x) &\equiv r_1(x) \pmod{f_1(x)} \\ f(x) &\equiv r_2(x) \pmod{f_2(x)} \\ &\vdots \\ f(x) &\equiv r_k(x) \pmod{f_k(x)} \end{aligned}$$

   Furthermore, the solution $f(x)$ is unique modulo the product $F(x) = f_1(x)f_2(x)\ldots f_k(x)$.

9. **Lagrange Interpolation Theorem**

   Let $a_1, \ldots, a_n \in \mathbb{F}$ be distinct numbers. For any given $b_1, \ldots, b_n \in \mathbb{F}$, there exists a unique polynomial $f \in \mathbb{F}[x]$ with $\deg f \leq n-1$ such that $f(a_i) = b_i$ for all i. Find this polynomial $f(x)$.

10. Assume $f(x)$ is a polynomial of degree n. If $f(k) = \frac{k}{k+1}$ for $k = 0, 1, \ldots, n$, find $f(n+1)$.

Section C Irreducible Polynomial

1. Let $u \in \mathbb{C}$. If there exists a non-zero polynomial $f(x)$ with rational coefficients such that $f(u) = 0$, then u is called an **algebraic number**.

   (1) Prove that $\sqrt{2}$ and $\sqrt{2} + \sqrt{3}$ are both algebraic numbers;

   (2) If u is an algebraic number, then there exists a unique monic polynomial $g(x)$ of minimal degree over the field of rational numbers such that $g(u) = 0$. This polynomial $g(x)$ is uniquely determined by u and is called the **minimal polynomial** of u.
   Prove: $g(x)$ is an irreducible polynomial over the field of rational numbers, and for any $f(x) \in \mathbb{Q}[x]$, the following holds:

$$f(u) = 0 \iff g(x) \mid f(x);$$

   (3) Find the minimal polynomial of the algebraic number $\sqrt{2} + \sqrt{3}$.

2. Let $p(x)$ be an irreducible polynomial over the field $\mathbb{F}$, and $f(x)$ be a polynomial over $\mathbb{F}$. Prove: If a complex root a of $p(x)$ is also a root of $f(x)$, then $p(x) \mid f(x)$. In particular, every complex root of $p(x)$ is a root of $f(x)$.

3. Let $f(x)$ be a polynomial with rational coefficients. Given that $\sqrt[n]{2}$ is a root of $f(x)$, prove that $\sqrt[n]{2}\varepsilon, \sqrt[n]{2}\varepsilon^2, \dots, \sqrt[n]{2}\varepsilon^{n-1}$ are also roots of $f(x)$, where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is an n-th root of unity.

4. Let $\mathbb{Q}(\sqrt[n]{2}) = \{a_0 + a_1\sqrt[n]{2} + a_2\sqrt[n]{4} + \cdots + a_{n-1}\sqrt[n]{2^{n-1}} \mid a_i \in \mathbb{Q}, 0 \le i \le n-1\}$. Prove that $\mathbb{Q}(\sqrt[n]{2})$ is a number field, and find a basis for $\mathbb{Q}(\sqrt[n]{2})$ as a vector space over $\mathbb{Q}$.

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5. Let $p(x) \in \mathbb{Q}[x]$ be an irreducible polynomial with odd degree $\deg(p(x)) > 1$. Let α_1 and α_2 be two distinct complex roots of $p(x)$. Prove that $\alpha_1 + \alpha_2$ is not a rational number.

6. Let $a_1, a_2, \dots, a_n$ be distinct integers. Prove that:

   (1) $f(x) = (x-a_1)(x-a_2)\dots(x-a_n) - 1$ is irreducible over $\mathbb{Q}$;

   (2) $g(x) = (x-a_1)^2(x-a_2)^2\dots(x-a_n)^2 + 1$ is irreducible over $\mathbb{Q}$.

7. Let $a_1, a_2, \dots, a_n$ be distinct integers, and let $f(x) = (x-a_1)(x-a_2)\dots(x-a_n) + 1$.

   (1) Prove that if n is odd or n is an even integer such that $n \ge 6$, then $f(x)$ is irreducible over $\mathbb{Q}$;

   (2) Is $f(x)$ irreducible over $\mathbb{Q}$ when $n = 2$ or $n = 4$?

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8. Let $f(x)$ be a polynomial with $\deg f(x) = n \ge 1$. If $f'(x) \mid f(x)$, prove that $f(x)$ has a root of multiplicity n.

9. Let $m \in \mathbb{N}$, prove:

   (1) $\displaystyle\prod_{k=1}^{m-1} \sin\frac{k\pi}{2m} = \frac{\sqrt{m}}{2^{m-1}}$

   (2) $\displaystyle\prod_{k=1}^{m-1} \cos\frac{k\pi}{2m} = \frac{\sqrt{m}}{2^{m-1}}$

   (3) $\displaystyle\prod_{k=1}^{m} \sin\frac{(2k-1)\pi}{2(2m+1)} = \frac{1}{2^m}$

   (4) $\displaystyle\prod_{k=1}^{m} \cos\frac{k\pi}{2m+1} = \frac{1}{2^m}$

   (5) $\displaystyle\prod_{k=1}^{m} \sin\frac{k\pi}{2m+1} = \frac{\sqrt{2m+1}}{2^m}$

   (6) $\displaystyle\prod_{k=1}^{m} \cos\frac{(2k-1)\pi}{2(2m+1)} = \frac{\sqrt{2m+1}}{2^m}$

   (7) $\displaystyle\prod_{k=1}^{m} \sin\frac{(2k-1)\pi}{4m} = \frac{\sqrt{2}}{2^m}$

   (8) $\displaystyle\prod_{k=1}^{m} \cos\frac{(2k-1)\pi}{4m} = \frac{\sqrt{2}}{2^m}$

10. Find the standard factorization of the following polynomials in $\mathbb{C}[x]$:

    (a) $x^{2n} + x^n + 1$;

    (b) $(x + \cos\theta + i\sin\theta)^n + (x + \cos\theta - i\sin\theta)^n$;

(c) $(x-1)^n + (x+1)^n$;

(d) $x^n - \binom{2n}{2}x^{n-1} + \binom{2n}{4}x^{n-2} + \cdots + (-1)^n\binom{2n}{2n}$;

(e) $x^{2n} + \binom{2n}{2}x^{2n-2}(x^2-1) + \binom{2n}{4}x^{2n-4}(x^2-1)^2 + \cdots + (x^2-1)^n$;

(f) $x^{2n+1} + \binom{2n+1}{2}x^{2n-1}(x^2-1) + \binom{2n+1}{4}x^{2n-3}(x^2-1)^2 + \cdots + \binom{2n+1}{2n}x(x^2-1)^n$.

Section D Multivariate Polynomial

1. Let $1 \le k \le n$. Prove that in $\mathbb{F}[x_1, x_2, \ldots, x_n]$, we have the following identities:

$$s_k = \begin{vmatrix} \sigma_1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2\sigma_2 & \sigma_1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 3\sigma_3 & \sigma_2 & \sigma_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 4\sigma_4 & \sigma_3 & \sigma_2 & \sigma_1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (k-1)\sigma_{k-1} & \sigma_{k-2} & \sigma_{k-3} & \sigma_{k-4} & \sigma_{k-5} & \sigma_{k-6} & \cdots & \sigma_1 & 1 \\ k\sigma_k & \sigma_{k-1} & \sigma_{k-2} & \sigma_{k-3} & \sigma_{k-4} & \sigma_{k-5} & \cdots & \sigma_2 & \sigma_1 \end{vmatrix}$$

and

$$\sigma_k = \frac{1}{k!}\begin{vmatrix} s_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ s_2 & s_1 & 2 & 0 & \cdots & 0 & 0 \\ s_3 & s_2 & s_1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ s_{k-1} & s_{k-2} & s_{k-3} & s_{k-4} & \cdots & s_1 & k-1 \\ s_k & s_{k-1} & s_{k-2} & s_{k-3} & \cdots & s_2 & s_1 \end{vmatrix}.$$

2. Solve the system of equations

$$\begin{cases} x_1 + x_2 + \cdots + x_n = n, \\ x_1^2 + x_2^2 + \cdots + x_n^2 = n, \\ \qquad \vdots \\ x_1^n + x_2^n + \cdots + x_n^n = n. \end{cases}$$

3. Let $A \in \mathbb{C}^{4\times 4}$ satisfy $\mathrm{Tr}(A^i) = i$ for $i = 1, 2, 3, 4$. Find the determinant $\det(A)$.

4. Find the power sums $s_1, s_2, \ldots, s_n$ of the roots of the polynomial

$$\begin{aligned} f(x) = x^n &+ (a+b)x^{n-1} + (a^2+ab+b^2)x^{n-2} + \ldots \\ &+ (a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})x \\ &+ (a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n). \end{aligned}$$

5. Find the power sums $s_1, s_2, \ldots, s_n$ of the roots of the polynomial:

$$f(x) = x^n + (a+b)x^{n-1} + (a^2+b^2)x^{n-2} + \cdots + (a^{n-1}+b^{n-1})x + (a^n+b^n).$$

6. Find the partial derivative $\frac{\partial \sigma_k}{\partial x_i}$ of the elementary symmetric polynomial $\sigma_k(x_1, x_2, \ldots, x_n)$ with respect to x_i, and calculate:

$$\sum_{i=1}^{n} \frac{\partial \sigma_k}{\partial x_i}.$$